Stochastic Geometry

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2019/2020

1 Abstract

Materials are heterogeneous and often have a complex internal microstructure. This microstructure determines most of the materials physical properties. Analysing that microstructure ususally involves analysing images obtained experimentally (e.g. through microscopy). Also generation of microstructures that reproduce geometric features is useful in computer based simulations of those materials. Stochastic Geometry revolves around tools for both analysing and simulating said microstructures.

2 Notation

In this excursion into Stochastic Geometry, the following notations will be used throughout

$$\begin{aligned}
\breve{B} &= \{-x, x \in B\} \\
A_x &= \{x + a | a \in A\} = A \oplus \{x\} \\
A^c &= \{x \notin A\} \\
\lambda A &= \underbrace{A \oplus \cdots \oplus A}_{\lambda \text{times}}
\end{aligned}$$

3 Mathematical Operators

The analysis of geometrical structures is theoretically described by mathematical morphology. It is a general framework that can be applied to more than just digital images and is fundamental in random set theory. Therefore it is practical to start with a few concepts from mathematical morphology.

3.1 Minkowski Operations

Definition 1:
Let A and B be subsets of $\mathbb{R}^d$. The Minkowski Addition is then defined by:

$$A \oplus B = \{a + b | a \in A, b \in B\} \tag{1}$$

Definition 2:
Let A and B be subsets of E. The Minkowski Substraction is then defined by:

$$A \ominus B = (A^c \oplus B)^c = \bigcap_{x \in B} A_x \tag{2}$$

3.2 Dilation and Erosion

Put in simple terms, the idea behind mathematical morphology is to analyse a set A by probing it with another compact set K, the structuring element. This makes use of classical set operator like union and intersection.

Definition 3:
Let A be a closed set in a topological space E. The dilated of the set A by the compact structuring element K is the set:

$$D^K(A) = \{x \in E | K_x \cap A \neq \emptyset\} \tag{3}$$

Similarly, the erosion is defined:

Definition 4:

$$E^K(A) = \{x \in E | K_x \subset A\} \tag{4}$$

These two operators are dual in the sense, that $D^K(A) = (E^K(A^c))^c$, the dilation of A is the complement to the erosion of the complement of A, wrt. the same structuring element. Using Minkowski operations, erosion and dilation can be written as:

$$E^K(A) = A \ominus \check{K} \tag{5}$$
$$D^K(A) = A \oplus \check{K} \tag{6}$$

3.3 Opening and Closing

Combining dilation and erosion, one can define two new morphological operators. Let $A, B \in \mathcal{P}(E)$ be subsets of a topological space E.

Definition 5:
The opening A_B and the closing A^B of the set A by B are defined by:

$$A_B = \left(A \ominus \check{B}\right) \oplus B \tag{7}$$
$$A^B = \left(A \oplus \check{B}\right) \ominus B \tag{8}$$

These operator alone provide a lot of functionality both in mathematical morphology as well as other applications such as image denoising. They are the fundamental building blocks of stochastic geometry.

There are some interesting properties of the opening respective the closing, here without proof:

- They are *idempotent*, that is $(A_B)_B = A_B$ and $(A^B)^B = A^B$.
- They are *increasing*, that is $A \subset A'$ implies $A_B \subset A'_B$ and $A^B \subset A'^B$.
- They are *dual* to each other: $(A^C)_B = (A^B)^C$ and $(A^C)^B = (A_B)^C$.
- The opening is *anti-extensive*: $A_B \subset A$.
- The closing is *extensive* $A \subset A^B$.

4 Granulometry

The first application of openings and closings in this context are the granulometry operators. These are families of openings (resp. closings) of increasing sizes, which allows to characterize size distributions of connected components of random sets. [1]

Definition 6:
A granulometry is formally defined as a family of set operators Φ_λ depending on a positive parameter λ such that:

1. *For all A in $\mathcal{F}(E)$, $\Phi_\lambda(A) \subset A$ — Φ_λ is anti-extensive*
2. *If $A \subset B$, then $\Phi_\lambda(A) \subset \Phi_\lambda(B)$ — Φ_λ is increasing*
3. *$\Phi_\lambda \circ \Phi_\mu = \Phi_\mu \circ \Phi_\lambda = \Phi_{max(\lambda,\mu)}$*

4.1 Granulometry by Opening

Definition 7:
Let K be a convex set then the following defines the granulometry by opening for all closed sets A of $\mathcal{F}(E)$:

$$\Phi_\lambda(A) = A_{\lambda K} = \left(A \ominus \lambda\check{K}\right) \oplus \lambda K \tag{9}$$

Beweis. To prove that a granulometry by opening is indeed a granulometry, all three criteria must be fulfilled. Now the second property is a direct consequence of the opening beeing an increasing operation. To prove the third property, one can without loss of generality assume:

$$\lambda = \nu + \mu$$

Then:

$$\Phi_\lambda(A) = (A \ominus \lambda\check{K}) \oplus K \tag{10}$$

$$= \underbrace{(A \ominus \lambda\check{K}) \oplus \nu K}_{C} \oplus \mu K \tag{11}$$

$$= C \oplus \mu K \tag{12}$$

Therefore $\Phi_\lambda(A)$ is open with respect to μK or in other words $\Phi_\mu(A) = A$. Utilizing the indepotence this yields:

$$\Phi_\mu(A) = A \tag{13}$$

$$(\Phi_\mu \circ \Phi_\lambda)(A) = \Phi_\lambda(A) \tag{14}$$

$$\Rightarrow \Phi_\lambda \circ \Phi_\mu \circ \Phi_\lambda = \Phi_\lambda \circ \Phi_\lambda = \Phi_\lambda \tag{15}$$

Now using the anit-extensiveness of the opening operation:

[1] Traditionally, granulometries or rather granular spectra were determined experimentally in grainy materials by sieving with increasingly fine sieves.

$$\Phi_\lambda(A) \subset (\Phi_\lambda \circ \Phi_\mu)(A) \tag{16}$$

But since the opening is an increasing operation, the converse is true as well:

$$(\Phi_\lambda \circ \Phi_\mu)(A) \subset \Phi_\lambda(A) \tag{17}$$

This imples both $\Phi_\lambda \circ \Phi_\mu = \Phi_\lambda$ and $\Phi_\lambda \subset \Phi_\mu$, the third and the first property. □

4.2 Granular Spectrum

If condition 1, the anti-extensiveness, is negated, the result is called an anti-granulometry. A family of set operators defined as:

$$\Phi_\lambda(A) = A^{\lambda K} = \left(A \oplus \lambda \check{K}\right) \ominus \lambda K \tag{18}$$

is such an anti-granulometry. More specifically it is the anti-granulometry by closing. Together, the granulometry and anti-granulometry form a *granular spectrum.*

5 Choquet Capacity and Covariance

The previously introduced concepts of mathematical morphology come in very handy when studying random sets. In particulat, when translating some compact set K in an observation window (e.g. a picture) to analyse a random set A, two elementary events can occur: $K \cap A = \emptyset$ and $k \cap A \neq \emptyset$. Therefore one can define a characterizing functional $T(K)$:

Definition 8:

$$T(K) = P\{A \cap K \neq \emptyset\} = 1 - P\{K \cap A^c\} = 1 - Q(K) \tag{19}$$

$T(K)$ is called the Choquet Capacity of A.

The choice of structuring element K is fundamental to the information obtained from the studied set A.

Example 1:
Let $K = \{x\}$ be a single point.

$$T(\{x\}) = P\{\{x\} \cap A \neq \emptyset\} = P\{x \in A\} \tag{20}$$

This is also called the spacial law of the set A.

5.1 Covariance

Definition 9:
The covariance of a random set $A \subset \mathbb{R}^n$ is the function C_A defined on $\mathbb{R}^n \times \mathbb{R}^n$ by:

$$C_A(x, x+h) = P\{x \in A, x+h \in A\}, \tag{21}$$

with $h \in \mathbb{R}^n$.

The covariance of the set A at a given point x for a distance h is the probability that both x and $x+h$ belong to A. Note that for $h = 0$ this just corresponds to the spacial law. Usually it is estimated from experimentally obtained samples, using the following equation, if A is stationary and ergodic:

$$C_A(h) = P\{x \in A \cap A_{-h}\} = V(A \cap A_{-h}) = V(A \ominus \check{H}) \tag{22}$$

where $H = \{-h, h\}$. The Covariance therefore gives information about the presence of periodic structures in the random set A.

References

[1] Bruno Figliuzzi, *Stochastic Geometry*, `http://www.cmm.mines-paristech.fr/~figliuzzi/Stochastic_Geometry.pdf`, Paris, 2015.

[2] Georges Matheron, *Random Sets and Integral Geometry*, John Wiley & Sons Inc., New York, 1975.

[3] Petros Maragos, *Pattern Spectrum and Multiscale Shape Representation*, Havard University, Cambridge, 1989.